Leslie Population Models in Predator-Prey and Competitive Populations

Predator-Prey Modelling Group (SMALL 2024)

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Lotka-Volterra Model

Continuous prey/predator densities $x(t), y(t) \in [0,1]$:

$$\frac{dx}{dt} = rx(1-x) - axy, \qquad \frac{dy}{dt} = ay(x-y). \tag{1}$$

Theorem (Merdan)

If $r - \alpha\beta > 0$, the system converges to the positive stable state

$$(x_*, y_*) = \left(\frac{r-\alpha\beta}{a+r}, \frac{r-\alpha\beta}{a+r}\right). \tag{2}$$

Assuming an homogenous predator and prey population, stability is guaranteed.

Leslie Matrices

We substitute the classical continuous model with finite-difference methods.

Definition (Simple Leslie matrix, fertility f)

$$L_f := \begin{bmatrix} f & f & \cdots & f & f \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad \vec{p}_n = L_f^n \, \vec{p}_0. \tag{3}$$

Example Model (Eqs. 1.2–1.7)

Whales (3 age groups $a_n^{(i)}$) consuming plankton b_n:

$$\vec{p}_{n+1} = \widetilde{L}\,\vec{p}_n = \begin{bmatrix} f & f & f & m \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -k & -k & -k & 1+F \end{bmatrix} \vec{p}_n, \quad \vec{p}_n = (a_n^{(1)}, a_n^{(2)}, a_n^{(3)}, b_n)^T. \tag{4}$$

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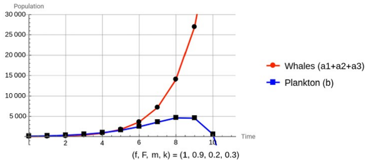
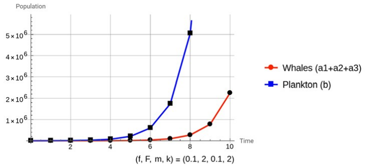


(A) High predation and prey exhaustion



(B) Low predation and mutual population growth

Predator-Prey Model (Def. 3.1)

Definition

Predator $\vec{\alpha}_n$, prey $\vec{\beta}_n \in \mathbb{R}^N_{\geq 0}$:

$$\vec{\alpha}_{n+1} = L_a \, \vec{\alpha}_n + km \, \vec{\beta}_n, \tag{5}$$

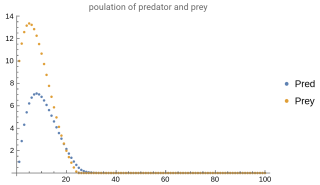
$$\vec{\beta}_{n+1} = L_b \, \vec{\beta}_n - k \, \vec{\alpha}_n. \tag{6}$$

k: predation ratio, m: nurturing ratio $(k, m > 0)$.

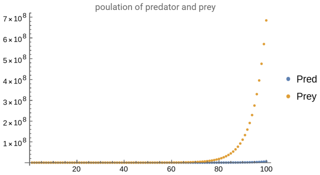
L_a: dominant eigenvalue < 1; L_b: dominant eigenvalue > 1.

Behavior

If consumption is too high, both populations die out.



Overconsumption: extinction



Small consumption: growth

Goal

Goal

A **necessary and sufficient** condition on k for population growth.

Lemma (Prop. 3.5)

$$\vec{\alpha}_n = (L_a + L_b)\vec{\alpha}_{n-1} - L_b L_a \, \vec{\alpha}_{n-2} - mk^2 \vec{\alpha}_{n-2}. \tag{7}$$

Scalar case $N = 1$: $l_a < 1, \; l_b > 1$.

Theorem 3.9 (1×1 case)

Theorem

Neither population vanishes as $n \to \infty$ iff

$$k \leq \sqrt{\frac{(1-l_b)(l_a-1)}{m}}. \tag{8}$$

Proof.

Companion matrix eigenvalue (discriminant ≥ 0):

$$\lambda = \frac{l_a + l_b}{2} + \frac{\sqrt{D}}{2}, \quad D = \frac{(l_a+l_b)^2}{4} - k^2 m - l_a l_b. \tag{9}$$

Non-extinction $\iff \lambda \geq 1 \iff k \leq \sqrt{(1-l_b)(l_a-1)/m}$. □

Theorem 3.11 ($L_a = \rho L_b$)

Theorem

For $L_a = \rho L_b$,

$$\vec{\alpha}_n = \Lambda_1^n \vec{v}_1 + \Lambda_2^n \vec{v}_2, \tag{10}$$

where Λ_1, Λ_2 *are matricies that are defined in terms of* L_a, L_b *and with* $\lambda_{\max}$ *the dominant eigenvalue of* L_b. *The dominant eigenvalue of* Λ_1, Λ_2 *are bounded over by*

$$\|\Lambda_{\max}\| = \frac{(\rho+1)\lambda_{\max} + \sqrt{(\rho+1)^2\lambda_{\max}^2 + 4mk^2}}{2}. \tag{11}$$

Open Question

The case $[L_a, L_b] \neq 0$ remains open.

Competitive Model (Def. 4.1)

Definition

Same growth matrix $L = L_f$ $(f \geq 1/N)$:

$$\vec{\alpha}_{n+1} = \max(L\vec{\alpha}_n - km\,\vec{\beta}_n,\ \vec{0}), \tag{12}$$

$$\vec{\beta}_{n+1} = \max(L\vec{\beta}_n - k\,\vec{\alpha}_n,\ \vec{0}). \tag{13}$$

$k, m \in (0,1)$: interaction and competitive-advantage ratios.

Theorem 4.2 (Last Species Standing)

Theorem

With $\vec{\alpha}_0 = (\alpha_0, \ldots, \alpha_0)$, $\vec{\beta}_0 = (\beta_0, \ldots, \beta_0)$, *let* $D := \alpha_0 - \sqrt{m}\,\beta_0$.

- $D > 0$: *β vanishes, α grows.*
- $D < 0$: *α vanishes, β grows.*
- $D = 0$: *both vanish or both grow.*

Approach: reduce to eigenvalues, use modulus as $n \to \infty$:

$$\Lambda_{1,2} = L \pm k\sqrt{m}\,I, \qquad \vec{\alpha}_n \approx \Lambda_1^n\,\vec{v}_1, \quad \vec{v}_1 = -\frac{D}{2\sqrt{m}}(1, \ldots, 1). \tag{14}$$

P. aurelia and *P. caudatum*

Figure: *Paramecium aurelia* (left), *Paramecium caudatum* (right). *Source: wikipedia.*

Goal

Fit population data of *P. aurelia* and *P. caudatum* to the competitive model and find underlying parameters.

Error Function

Definition (Error of the fit)

Let T be the number of discrete timesteps in the experimental data, and $\overline{\alpha}_t, \overline{\beta}_t$ be the total population of respective species predicted by the competitive model. Then,

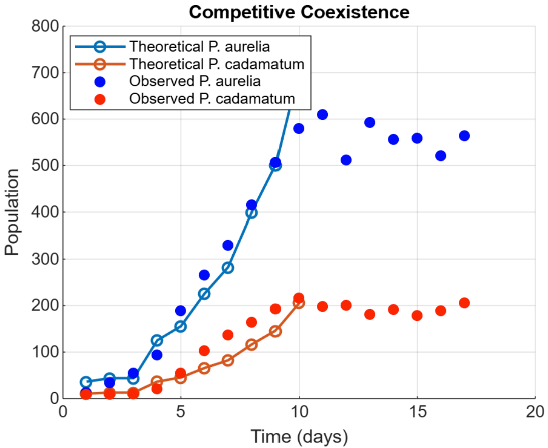
$$\chi(f, k, \beta_0, m) \ := \ \left(\sum_{t \ \leq \ T} \left(\frac{\overline{\alpha}_t - p(t)}{p(t)} \right)^2 + \sum_{t \ \leq \ T} \left(\frac{\overline{\beta}_t - q(t)}{q(t)} \right)^2 \right). \quad (15)$$

Machine Learning Scheme

Machine Learning Scheme

1. Evaluate χ for random choice of the four parameters. Among the random tuples, choose the tuple that minimizes the value of χ.
2. Perform a general gradient descent involving all four parameters, starting from the point chosen in step 1. Find the admissible value of k, m.
3. Define a subroutine, $\mathrm{Opt_g}$, that performs a gradient descent solely on parameter g with all other parameters fixed.
4. Finally perform an optimized gradient descent on parameter m. In the beginning of each loop, invoke the subroutine $\mathrm{Opt_g}$ to adjust the parameter g.

Admissible Fit



$$\chi = 3.11 < 10.351 \tag{16}$$

Open Problems

1. Can we provide a sufficient condition for population growth of a Predator-Prey model where $[L_a, L_b] \neq 0$?
2. Is it possible to create a Leslie matrix model that displays saturation?
3. Formulate a Leslie Population Model of open systems with migration.

Thank You!

Questions & Answers

Paper

Status: *Accepted for publication to the Fibonacci Quarterly.*

arXiv: `https://arxiv.org/abs/2412.19831`

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